Some Ideas from the Birkar-Cascini-Hacon-McKernan Proof

MMP Learning Seminar

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F 9/17/21

Main Theorem

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)) \text{ is finitely generated.}$$

Goal

(X, Δ) : projective klt pair

Δ big

$K_X + \Delta$ pseudoeffective

Then $K_X + \Delta$ has log terminal model.

Remark

Authors		
log terminal model	=	minimal model
canonical model	=	ample model

Important terms

Our setting

X: normal variety

$D = \sum d_i D_i$: $\mathbb{Q}$-divisor on X

D_i: distinct, irreducible

$K_X + D$: $\mathbb{Q}$-Cartier

$f : Y \longrightarrow X$ proper birational morphism

the discrepancy of E w.r.t. (X, D)

Can write: $$K_Y = f^*(K_X + D) + \sum_{\substack{E \subset Y \\ \text{prime divisor}}} a(E, D)\, E$$

discrepancy of (X, D)

$$\operatorname{discrep}(X, D) := \inf_E \{ a(E, D) \text{ s.t. } E \text{ is an exceptional divisor over } X \}$$

Assume D is a boundary (i.e., coeff.s of $D \in [0, 1]$).

Singularities of Pairs

Say (X, D) has

[Type of sing.s] singularities if discrep(X, D) is

klt	> -1 and $\lfloor D \rfloor = 0$
plt	> -1
dlt	≥ -1, $\text{center}_X E \subset$ non s.n.c. of (X, D)
lc	≥ -1

dlt: If $a(E, D) = -1$, then $\text{center}_X E \subseteq$ the s.n.c. locus of (X, D).

(X, D) is klt / plt / dlt / lc if it has klt / plt / dlt / lc singularities.

Note $\quad$ klt $\subsetneq$ plt $\subsetneq$ dlt $\subsetneq$ lc

(X, D) is <u>dlt</u> iff there exists a Zariski open $\mathcal{U} \subseteq X$ s.t.

(1) $\mathcal{U}$ smooth, $D|_{\mathcal{U}}$ is a s.n.c. divisor.

(2) Any log canonical center of (X, D) intersects $\mathcal{U}$ and is given by strata $\lfloor D \rfloor$.

The <u>log discrepancy</u> is $\alpha_E(X, D) := 1 + a(E, D)$

<u>Note</u>

$a(E,D) > 0 \Rightarrow \alpha_E(X,D) > 1$

$a(E,D) = 0 \Rightarrow \alpha_E(X,D) = 1$

$a(E,D) = -1 \Rightarrow \alpha_E(X,D) = 0$ $\rightsquigarrow$ E is a <u>log canonical place</u>.

(X, D)

A subvariety $V \subseteq X$ is a <u>log canonical center</u> if there exists a

- proper birational morphism $\mu : Y \to X$
- prime divisor E on Y

with discrepancy $\underbrace{a(E, D) = -1}$ s.t. $\mu(E) = V$.

so $\alpha_E(X, D) = 0$ (i.e., E is a lc place)

i.e., a log canonical center is the image of a lc place.

A <u>log resolution</u> of (X, D) is a proper birational morphism $f : Y \to X$ s.t.

1) $\mathrm{Exc}(f)$ is a divisor $E = \sum E_i \subset Y$; E_i's : irreducible components
2) Y nonsing.
3) $\mathrm{supp}(f^{-1}(D) \cup E)$ is a snc divisor.

Ex C : Cone over an elliptic curve

C

$$\varphi^*(K_C) = K_Y + cE$$

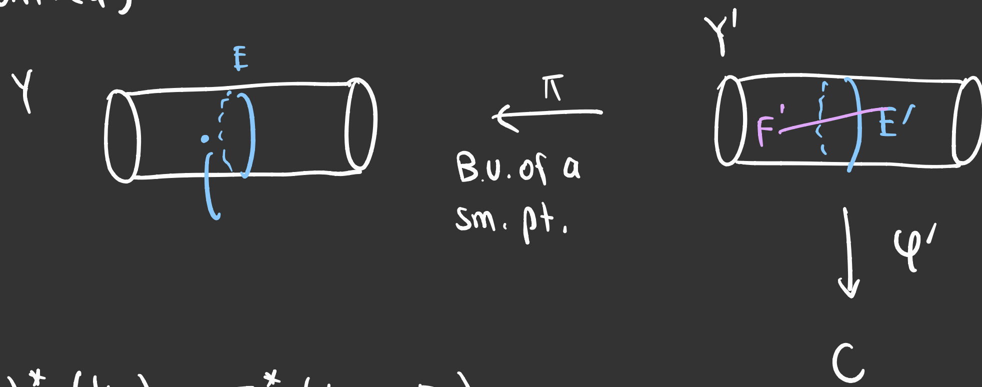
adj.

$$\varphi^*(K_C) = K_Y + E$$

$\Rightarrow \alpha_E(X, D) = 1 + (-1) = 0$ $\quad a(E, D) = -1$

$\Rightarrow$ E is an lc place.

Ex (cont.ed)

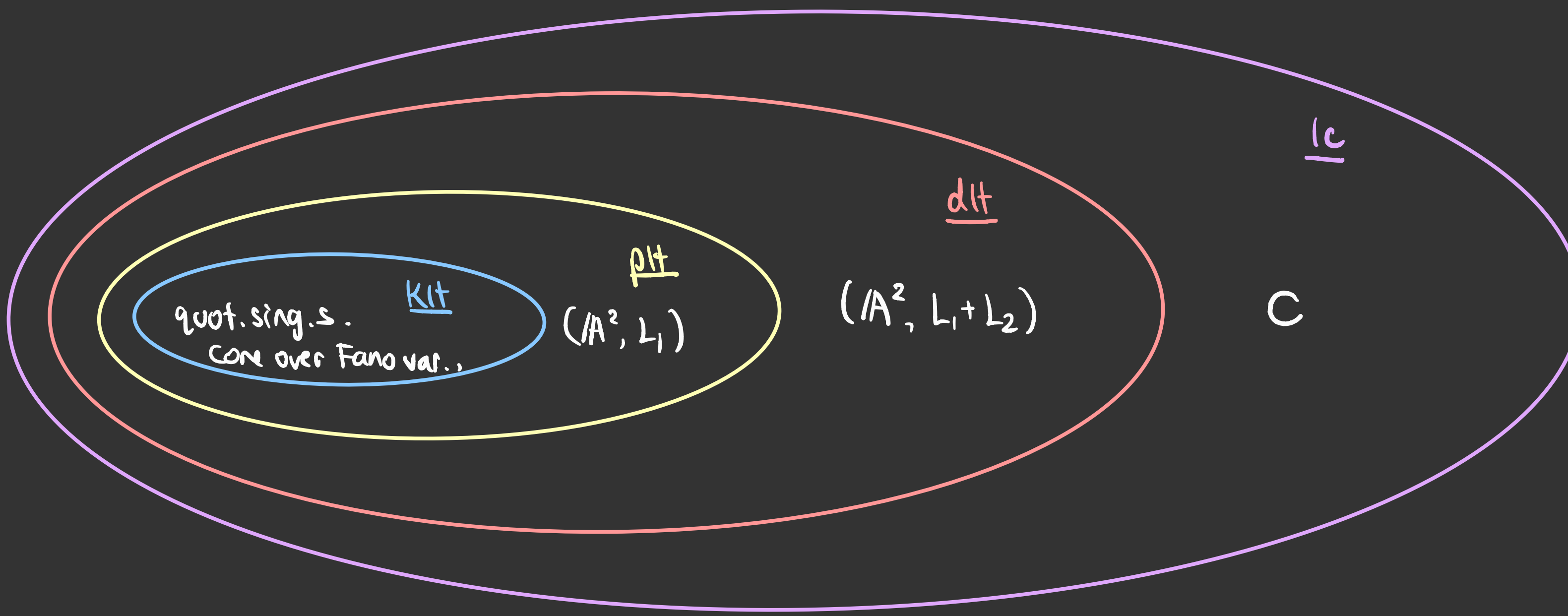


$$(\varphi')^*(K_C) = \pi^*(K_Y + E)$$
$$= \pi^*(K_Y) + \pi^*(E)$$
$$= (K_{Y'} - F') + (E' + F')$$
$$= K_{Y'} + E' + 0F'$$

$$\alpha_F(X, D) = 1 > 0$$

$\Rightarrow$ F' is not a lc place.

klt $\subsetneq$ plt $\subsetneq$ dlt $\subsetneq$ lc

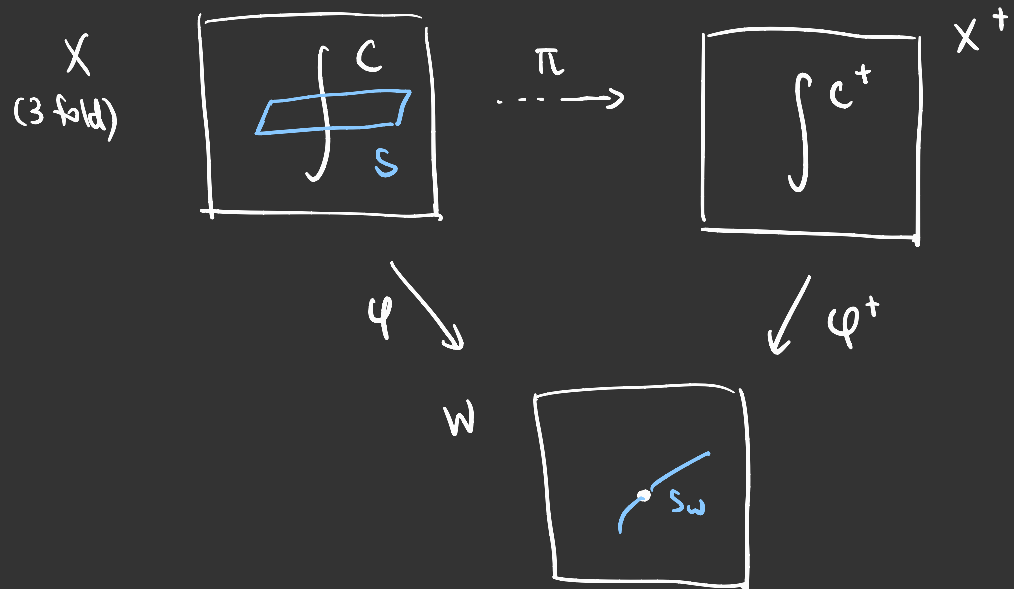


$f: X \to W$ a projective birational morphism, $\underbrace{\text{small}}_{\text{Exc}(f) \text{ has codim} \geq 2 \text{ in } X}$

(X, D): plt with $S = \lfloor D \rfloor$ irred.

f is a <u>pl-flipping contraction</u> if $\rho(X/W) = 1$ and $-(K_X + D)$ is ample over W.

<u>Picture</u>



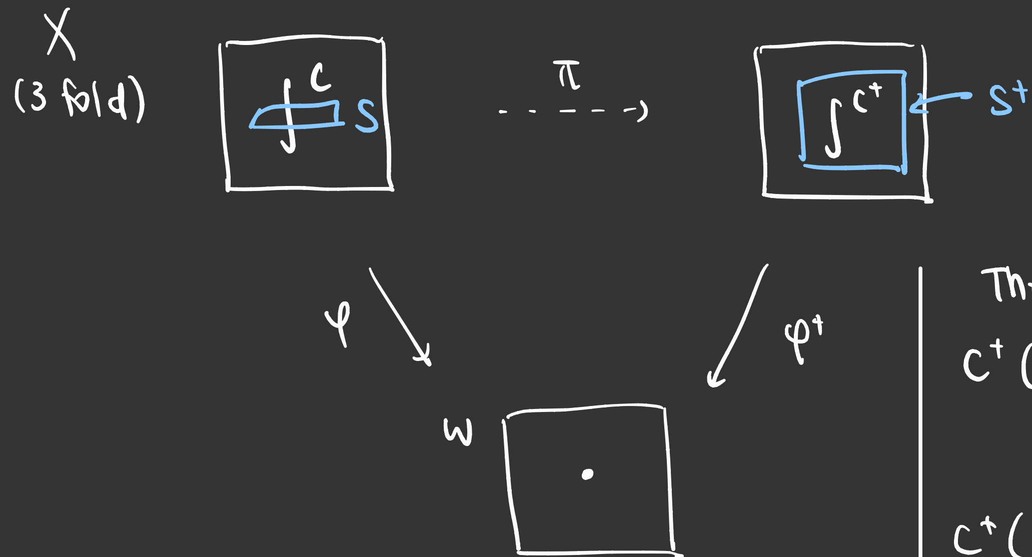
If we replace

$$(X, \Delta) \rightsquigarrow (X, \Delta + S) \text{ where } S = \pi^* S_w :$$

$(X, \Delta + S)$ plt $\Rightarrow$ we still have a flip.

Moreover, if $S = \pi^* S_w$

$$\text{flip}(X, \Delta + S) \equiv \text{flip}(X, \Delta)$$



Suppose $S \cdot C > 0$. WTS $S^+ \cdot C^+ < 0$.

Pf

$\exists\, \alpha > 0$ s.t. $(K_X + \Delta + \alpha S) . C = 0$.

Cone Thm $\Rightarrow K_X + \Delta + \alpha S = \varphi^* L$

$$\Rightarrow \underbrace{\pi_*(K_X + \Delta + \alpha S)}_{\pi_* \varphi^* L \,=\, (\varphi^+)^* L} = K_X + \Delta^+ + \alpha S^+$$

Then

$$C^+ (K_{X^+} + \Delta^+ + \alpha S^+) = (\varphi^+)^* L \cdot C^+$$
$$= L \cdot [0] = 0$$

$$C^+ (K_{X^+} + \Delta^+) > 0$$

So $\alpha S^+ \cdot C^+ < 0$

$\Rightarrow S^+ \cdot C^+ < 0$. $\square$

Why are pl-flips important?

Recall WTS $\bigoplus H^0(m(K_X+\Delta))$ is f.g.

$$
\begin{tikzcd}
e(X/W)=1: & \bigoplus H^0(m(K_X+\Delta+S)) \arrow[d] & (**)\ \text{f.g.} \\
e(S) & \bigoplus H^0(m(K_S+\Delta_S)) & (*)\ \text{f.g.} \arrow[u, Rightarrow]
\end{tikzcd}
$$

Simplified Versions of Theorems Used

Thm A (Existence of pl-flips)

(X, Δ) plt

$f : X \to Z$ pl-flipping contraction

Then the flip $\pi : X \dashrightarrow X^+$ exists.

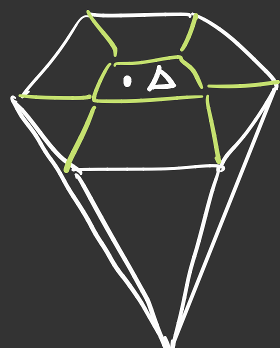
Motivation for Thm B

Want: finiteness of ample models.

- Start with $K_X+\Delta$
- Consider all perturbations of Δ

} give

polyhedra



$K_X+\Delta+\varepsilon D$: stabilize at some point.

Note only those that keep $K_X+\Delta$ klt.

By finiteness of ample models:

As we perturb Δ, get finitely many ample models, i.e.,

finitely many of $\operatorname{Proj}\left(\bigoplus_{m\geq 0} H^0(m(K_X+\Delta))\right)$

$\rightsquigarrow$ chamber decomposition where Proj(elts in same chamber) are equal

Thm B (special Finiteness)

(X, Δ) klt

$(X, \Delta+S)$ plt $\qquad$ S = sum of finitely many prime divisors

$\lfloor \Delta + S \rfloor = S$: irred

Then there are finitely many ample models for perturbations of $K_X+\Delta+S$, so that any other ample model of a perturbation of $K_X+\Delta+S$ is isomorphic to one of the previous ones around S.

Thm C (Existence of log terminal models)

(X,Δ) klt

$K_X+\Delta$ big

Then there exists a MMP that terminates with a log terminal model,

i.e., $\exists$ a $(K_X+\Delta)$-negative contraction $(X \dashrightarrow X_{min})$ s.t.

$$K_{X_{min}} + \Delta_{min} \text{ is nef},$$

$$\text{where } \Delta_{min} = \varphi_* \Delta.$$

Fact $K_X+\Delta$ big $\rightsquigarrow$ Δ big

Trick:
$$(1+\varepsilon)(K_X+\Delta) = K_X+\Delta+\underbrace{\varepsilon(K_X+\Delta)}_{\equiv\, H \geq 0}$$

Fact big $\Longrightarrow$ effective

Thm D (Nonvanishing Thm)

(X, Δ) klt

Δ big

$K_X + \Delta$ pseudoeffective

Then $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$.

Thm E (Finiteness of Models)

The global version of Thm B.

Thm F

$K_X + \Delta$ $\mathbb{Q}$-Cartier $\Rightarrow$ $R(X, K_X + \Delta)$ is f.g.

(X, Δ) : klt

$$R(X, K_X+\Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X \lfloor m(K_X + \Delta) \rfloor))$$

Some main ideas of proof

$K_X + \Delta$ big

WTS: $R(X, K_X + \Delta)$ is f.g.

Approach Construct a log terminal model for X.

$$\begin{bmatrix}\text{Existence and term.}\\ \text{of flips in dim } (n-1)\end{bmatrix} \Rightarrow \begin{bmatrix}\text{Existence of pl-flips}\\ \text{in dim. } n\end{bmatrix} \Rightarrow \begin{bmatrix}\text{Existence of klt-flips}\\ \text{in dim. } n\end{bmatrix}$$

Hacon and McKernan '05

WMA pl-flips in dim n exist

Suppose $K_X + \Delta$ big.

Run MMP in dim. n.

$$\rightsquigarrow (X, \Delta) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_{min}, \Delta_{min})$$

$$K_{X_{min}} + \Delta_{min} \text{ big and nef } \underset{\substack{\text{Kollár}\\ \text{(Basept free thm)}}}{\Longrightarrow} K_{X_{min}} + \Delta_{min} \text{ } \underbrace{\text{semiample}}_{\text{so } \exists \text{ morphism } X_{min} \to X_{can}}$$

$$R(X, K_X + \Delta) \cong R(X, K_{X_{min}} + \Delta_{min}) \cong R(X, K_{can} + \Delta_{can})$$

MMP with scaling

$K_X + \Delta$ s.t. (X, Δ) klt

A: ample divisor

$K_X + \Delta + \lambda A$ nef for some $\underbrace{\lambda \geq 0}_{\text{choose minimal such } \lambda}$

If $\lambda = 0$ stop; since $K_X + \Delta + \lambda A$ is nef.

Otherwise There exists a $(K_X+\Delta)$-extemal ray R s.t. R is a $(K_X+\Delta+\lambda A)$-trivial ray.

Contraction associated to ray is a $(K_X+\Delta)$-flip $\equiv$ $(K_X+\Delta+\lambda A)$-flop

$\Rightarrow$ X is a min. model for $K_X+\Delta+\lambda A$.

Idea of MMP with scaling

Do flips :

$$\underbrace{(X, \Delta)}_{\substack{\text{min. model for} \\ K_X + \Delta + \lambda A}} \dashrightarrow \underbrace{(X_1, \Delta_1)}_{\substack{\text{min mod for} \\ K_{X_1} + \Delta_1 + \lambda A_1}} \dashrightarrow \cdots \dashrightarrow (X_j, \Delta_j)$$

Eventually, λ is no longer the min value s.t. $K_{X_j} + \Delta_j + \lambda A_j$ is nef.

λ decreases say $\lambda \rightsquigarrow \lambda'$

⋮ (Repeat)

Process: MMP with scaling

Quasi MMP with scaling

$(X, \Delta + S)$: plt

Want S: $0 \leq s \sim aK_X + \Delta$

Then $K_X + \Delta + S|_S = K_S + \Delta_S$

Consider 2 sequences of flips A and B :

B $(X, \Delta + S) \dashrightarrow (X, \Delta_1 + S_1) \dashrightarrow \cdots$ B terminates around S

$\Uparrow$ B.C.H.M.

A $(S, \Delta_S) \dashrightarrow (S_1, \Delta_{S_1}) \dashrightarrow \cdots$ If A terminates

(S, Δ_S) klt $\Rightarrow \exists$ finitely many divisors over (S, Δ_S) w/ log discrepancy in $(0,1)$

Then blowups in A eventually terminate.

Want C s.t.

- $K_X + \Delta \sim_{\mathbb{Q}} D + \alpha C$
- $K_X + \Delta + C$ is dlt, nef
- $\operatorname{Supp}(D) \subseteq \underbrace{\lfloor \Delta \rfloor}_{S}$

The point:

R: extreme ray $\quad (K_X+\Delta).R < 0$

$K_X + \Delta + C$ nef and $C.R \geq 0 \Rightarrow (K_X+\Delta+C).R \geq 0$

So $(K_X+\Delta).R < 0 \Rightarrow (D+\alpha C).R \leq 0 \quad$ (also $D.R < 0$)

$\Rightarrow [R] \subseteq \operatorname{Supp}(D)$

<u>Done</u> $D \subseteq S$ so every $(K_X+\Delta)$ negative curve lies in S.

$\Rightarrow$ a min. model for $K_X+\Delta$ around S is a min mod for $K_X+\Delta$.

Then construct $D+\alpha C$ s.t. $K_X+\Delta \sim_{\mathbb{Q}} D+\alpha C$

Note Since $K_X+\Delta$ is big, we have $K_X+\Delta = A+E$ (A is ample)

Write $D = D_1 + D_2$: $D_1 \subseteq S$, $D_2 \not\subseteq S$

Do induction on # components of D_2.

If $D_2 = \emptyset$, then $D \subseteq S$ and done by previous page

If $D_2 \neq \emptyset$: done by induction and adjunction.

This finishes proof if $K_X+\Delta \sim_{\mathbb{Q}} D \geq 0$

$\square$

Sketch of Effectivity

$K_X + \Delta$ pseudoeff. and Δ big $\Rightarrow K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$ $(\dagger)$

If $K_X + \Delta$ big, then conclusion of $(\dagger)$ holds.

For any ample H,

$$\underbrace{h^0(X, \mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor + H))}_{\text{If this is } \leq c \text{ for all } m, \text{ then } K_X+\Delta \equiv N_\sigma(K_X+\Delta) \geq 0.} \text{ is a bdd fn. of } m.$$

i.e., $K_X + \Delta$ has no positive part in the Nakayama decomposition

$\hookrightarrow \underbrace{N_\sigma}_{\text{neg}} + P_\sigma$

Next trick:

produce a lc center for $K_X + \Delta + \frac{H}{m}$

Then: do adjunction to the lc center: $K_X + \Delta + \frac{H}{m}\Big|_S = K_S + \Delta_S + \frac{H_S}{m_S}$.

Idea of next steps

- Show $K_S + \Delta_S$ is pseudoeffctive w/ Δ_S big
- S has lesser dim. because its a lc center

$\Rightarrow$ $K_S + \Delta_S$ has a section

lift to a section of $K_X + \Delta$ using Kawamata-Viehweg Vanishing:

If $\Theta - (K_X + D)$ is big and nef, then $h^1(\Theta) = 0$.

take $\Theta = m(K_X + \Delta) - S$.

References

1. Flips for 3-folds and 4-folds, by Alessio Corti.

2. Classification of Higher Dimensional Algebraic Varieties, by Christopher Hacon and Sándor Kovács.

3. Singularities of the Minimal Model Program, by János Kollár.

4. Birational Geometry of Algebraic Varieties, by János Kollár and Shigefumi Mori

Acknowledgements

Thanks to Joaquín Moraga!